Topology and its Applications 24 (1986) 143-145
North-Holland

ARE THE BORROMEAN RINGS *A-B*-SLICE*

Michael H. FREEDMAN
Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, USA

Received 22 April 1985

A generalization of the slice problem for links is presented. The general 4-dimensional, topological surgery problem may be recast in this light.

AMS(MOS) Subj. Class.: 57M25, 57Q45, 57S25

surgery	topological	link theory
4-dimensional	group actions	Borromean rings

The preceding note [1] establishes an equivalence between topological 4-dimensional surgery and an extension problem for actions on S^3. Existence of (admissible) extensions implies a curious property, which we call *A-B*-slice, for the Borromean rings.[1] Possibly, someone can show that the Borromean rings are not *A-B*-slice. If so, the 4-dimensional surgery theorem (and the 5-dimensional proper *s*-cobordism theorem) is contradicted.

Define a *decomposition* of B^4 to be a pair of smooth, compact codimension-0 submanifolds with boundary $A, B \subset B^4$ satisfying:

(1) $A \cup B = B^4$,

(2) $A \cap B = \partial^- A = \partial^- B$,

(3) $\partial A = \partial^+ A \cup \partial^- A$, $\partial^+ A = \partial A \cap \partial B^4$, $\partial^+ A \cap \partial^- A =$ the Clifford torus $S^1 \times S^1 \subset S^3 = \partial B^4$, and similarly

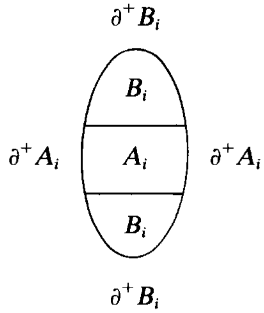
(3′) $\partial B = \partial^+ B \cup \partial^- B$, $\partial^+ B = \partial B \cap \partial B^4$, $\partial^+ B \cap \partial^- B = \partial^+ A \cap \partial^- A$.

Roughly put, a decomposition of B^4 is some extension to B^4 of the standard genus = 1 Hegard decomposition of $S^3 = \partial^+ A \cup \partial^+ B$.

Suppose $L \subset S^3$ is a tame link of l components with tubular neighborhood $\mathscr{N}$. Let $D(L)$ be the $2l$-component link obtained by pushing off an untwisted (i.e., linking number equal to zero) parallel to L. We say L is *A-B-slice* if there exist l decompositions: $(A_1, B_1), \ldots, (A_l, B_l)$ of B^4 and $2l$ self-homeomorphisms of B^4 $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_l$, such that the entire collection: $\alpha_1 A_1, \ldots, \alpha_l A_l, \beta_l B_1, \ldots, \beta_l B_l$ are pairwise disjoint and satisfy the boundary data: $\alpha_i \partial^+ A_i$ is a tubular neighborhood of the ith component of L and $\beta_i \partial^+ B_i$ is a tubular neighborhood of the ith component of the parallel copy of L, $1 \leq i \leq l$.

* Supported in part by NSF DM583-03126 and the MacArthur Foundation.
[1] And also for any iterated-ramified Bing double of the Hopf link.

Observation 1. If L is slice (in the topologically flat sense) then L is A-B-slice. Simply choose the decompositions (A_i, B_i) so that $(A_i, \partial^+ A_i)$ is a 2-handle and $(B_i, \partial^+ B_i)$ is a product collar $\cong(\partial^+ B_i \times I, \partial^+ B_i)$.



Observation 2. If the linking number between two components is non-zero, $\text{Link}(L_i, L_j) \neq 0$, then L is not A-B-slice. Let $a_i \subset \partial^+ A_i$ and $b_i \subset \partial^+ B_i$ be the core circles of the solid tori. Considered as cycles, it is impossible for both a_i to bound a rational 2-chain in A_i and b_i to bound a rational 2-chain in B_i; this would compute $0 = \text{link}(a_i, b_i) = 1$. On the other hand, if one (say a_i) does not bound, then the other must, for we have:

$$\begin{array}{ccc} & & H_2(B, Q) \\ & & \downarrow \beta \\ H_1(A_i; Q) & \xrightarrow[\cong]{\alpha = \text{Alexander duality}} & H_2(B_i, \partial^+ B_i; Q) \\ \cup\!\!\!\shortmid & & \downarrow \partial \\ 0 \neq [a_i] & & H_1(\partial^+ B_i; Q) = \langle [b_i] \rangle \end{array}$$

Note that $\alpha[a_i]$ links $[a_i]$ in B^4 so it cannot be in the image of β. Thus, $\partial\alpha[a_i]$ is non-trivial and hence a multiple of $[b_i]$. Now suppose L is A-B-slice with respect to the decompositions $(A_1, B_1), \ldots, (A_l, B_l)$. For all $i = 1, 2, \ldots, l$, there is exactly one side A_i or B_i, in which the core circle (a_i or b_i) bounds a rational 2-chain. The imbeddings $\alpha_i|_{A_i}$ or $\beta_i|_{B_i}$ carry these 2-chains disjointly into B^4. Adjusting for the slight difference between components of L and its parallel, all components of L bound disjoint rational 2-chains, so all pairwise linking numbers are zero, link $(L_i, L_j) = 0$.

Proposition. *The 4-dimensional topological surgery 'theorem', if true, implies that the Borromean rings are A-B-slice.*

Proof. In the notation of the preceding note and by a similar shrinking argument, $\hat{\mathcal{S}}$(Wh(Borromean Rings)) is homeomorphic to S^3. Thus there is an admissible action of F_3 on S^3 with fundamental domain $\Delta = S^3 - (D(\text{Borromean Rings}))$ for $\omega|_{\Omega_\omega}$. Our assumption of surgery and Theorem 2 (of the preceding article [1]) gives the admissible extension $\bar{\omega}: F_3 \to \text{homeo}^+(B^4)$. The quotient $\Omega_{\bar{\omega}}/\bar{\omega} = N^4$ may be cut open along topologically flat 3-manifolds (use Top transversality [3]) to obtain a manifold fundamental domain $\bar{\Delta}$ for $\bar{\omega}|_{\Omega_{\bar{\omega}}}$ with $\bar{\Delta} \cap \partial B^4 = \Delta$.

The closed complement $\text{cl}(B^4 - \bar{\Delta})$ has six connected components which may be labelled $A'_1, A'_2, A'_3, B_1, B_2$, and B_3 so that the natural generators x_1, x_2, and x_3 satisfy $\bar{\omega}x_i(A'_i) = \text{cl}(B^4 - B_i) \overset{\text{def}}{=} A_i$ for $i = 1, 2, 3$. Since every component of $\text{cl}(S^3 - \Delta)$ is an unknotted solid torus, (A_i, B_i), $i = 1, 2, 3$, are (topological) decompositions of B^4. Notice that $((\partial A'_1 \cup \partial A'_2 \cup \partial A'_3) \cup (\partial B_1 \cup \partial B_2 \cup \partial B_3)) \cap \partial B^4$ is simply $\text{cl}(S^3 - \Delta)$ with $(\partial A'_1 \cup \partial A'_2 \cup \partial A'_3) \cap \partial B^4$ being a tubular neighborhood of the Borromean rings and $(\partial B_1 \cup \partial B_2 \cup \partial B_3) \cap \partial B^4$ a disjoint tubular neighborhood of the parallel copy.

Taking $\alpha_i = \bar{\omega}x_i$ and $\beta_i = \text{id}_{B^4}$, $i = 1, 2, 3$, nearly completes the construction of the A-B-slices. The remaining point is that the decompositions of B^4, (A_i, B_i), $i = 1, 2, 3$, are into codimension zero topological submanifolds rather than smooth. To obtain smooth decompositions, simply approximate the intersection $A_i \cap B_i = \partial^- B_i$ relative to a neighborhood of $\partial(\partial^- B_i)$ (where $\partial^- B_i$ will already be smooth if transversality is applied carefully in the construction of $\bar{\Delta}$) by a smooth submanifold to obtain a new decomposition $(\tilde{A}_i, \tilde{B}_i)$ with $\partial^- \tilde{B}_i$ near $\partial^- B_i$. By compactness, the six subsets $A'_1, A'_2, A'_3, B_1, B_2, B_3 \subset B^4$ have some minimum separation $\varepsilon > 0$, thus using continuity of α_i^{-1}, δ-approximate $(\tilde{A}_i, \tilde{B}_i)$ to (A_i, B_i), $i = 1, 2, 3$, so as to preserve the pairwise disjointness of the six subsets; $i = 1, 2, 3$.

Conversely, the existence of A-B-slices for the Borromean rings implies that the whitehead double Wh(Bor) is Top-flat slice with π_1 (slice complement) freely generated by meridians (see [1]).

It follows from [2] and a little decomposition space theory that the elementary Whitehead link is A-B-slice. This is noteworthy since the Whitehead link is not slice (and in fact does not even bound disjoint maps of disks into B^4).

It appears to be a routine application of Donaldson's theory to show that the Borromean rings do not satisfy the smooth category analogy of A-B-slice condition. □

References

[11] Michael Freedman, A geometric reformulation of four dimensional surgery, Topology 24 (1985) (this issue) 135-143.

[2] Michael Freedman, Wh_3, to appear.

[3] Frank Quinn, Ends of Maps, III; Dimensions 4 and 5, J. Diff. Geom. 81 (1982) 503-521.